

RELATIONS BETWEEN EQUATIONS OF MUKAI VARIETIES

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ABSTRACT. This note is an answer to a problem proposed by Iliev and Ranestad. We prove that the projection of general nodal linear sections of suitable dimension of the Mukai varieties M_g are linear sections of M_{g-1} .

1. INTRODUCTION

In [6] Mukai gave a description of canonical curves, K3 surfaces and Fano threefolds of genus $g \leq 10$ in terms of linear sections of appropriate varieties. The description may be summarized in the following table

genus	Anticanonical model of a general Fano threefold
2	$X_6 \subset \mathbb{P}(1^4, 3)$
3	$X_4 \subset \mathbb{P}^4$
4	$X_{2,3} \subset \mathbb{P}^5$
5	$X_{2,2,2} \subset \mathbb{P}^6$
6	$X_{1,1} \subset Q_2 \cap G(2, 5) =: M_6^5$
7	$X_{1,1,1,1,1,1,1} \subset SO(5, 10) =: M_7^{10}$
8	$X_{1,1,1,1,1} \subset G(2, 6) =: M_8^8$
9	$X_{1,1,1} \subset LG(3, 6) =: M_9^6$
10	$X_{1,1} \subset \mathbb{G}_2 =: M_{10}^5$

In the table we use the notation X_{i_1, \dots, i_n} for the generic complete intersection of given degrees. The variety Q_2 is a generic quadric hypersurface. The notation $G(2, n)$ stand for the Grassmannians of planes in their Plücker embeddings. The variety $SO(5, 10)$ is the orthogonal Grassmannian parameterizing linear spaces of dimension 4 in a smooth

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eight dimensional quadric hypersurface in \mathbb{P}^9 in its spinor embedding. The variety $LG(3, 6)$ is the Lagrangian Grassmannian, it is a linear section of $G(3, 6)$ in its Plücker embedding parameterizing 3 spaces isotropic with respect to a chosen generic symplectic form. The variety G_2 is a linear section of $G(5, 7)$ in its Plücker embedding parameterizing 5 spaces isotropic with respect to a chosen generic four-form. The notation M_g and the name Mukai varieties has become common in this context.

As it was observed in [10, 9] the above description suggests that the projection of any general nodal linear section of M_g is a linear section of M_{g-1} . This is clearly true for the analogical problem formulated for cases of degree ≤ 6 as these are standard examples of Kustin-Miller unprojections (see [12]).

The case $g = 9$ was proved in [9] as the following theorem.

Theorem 1.1. *Let L be a general nodal hyperplane section of $LG(3, 6)$. Then the projection of L from the node is a linear section of $G(2, 6)$, containing a 6 dimensional quadric. Conversely each 7 dimensional linear section of S_{10} that contains a 6 dimensional quadric arises in this way.*

The proof followed from the construction of an appropriate bundle on the resolution of a nodal hyperplane section of $LG(3, 6)$.

In this note we reproof 1.1 together with the remaining cases in purely algebraic terms by simple analysis of equations of considered varieties.

This result may be used to construct an appropriate cascade of geometric transitions as described in [2]. Moreover, similarly as in [9] we describe in terms of the dual variety spaces of vector bundles on curves, K3 surfaces giving embeddings into suitable homogeneous varieties.

2. STATEMENTS

Theorem 2.1. *Let L be a general nodal 5 dimensional linear section of S_{10} . Then, the projection of L from the node is isomorphic to $G(2, 5) \cap Q$, where Q is a quadric in \mathbb{P}^9 .*

Theorem 2.2. *Let L be a general nodal hyperplane section of $G(2, 6)$. Then, the projection of L from the node is a linear section of S_{10} , containing a 6 dimensional quadric.*

Theorem 2.3. *Let L be a general nodal hyperplane section of G_2 . Then the projection of L from the node is a linear section of, $LG(3, 6)$, containing a 3 dimensional quadric.*

The proof of Theorems 2.1, 2.2, 1.1 and 2.3 is based on analyzing equations of the involved varieties.

3. EQUATIONS OF MUKAI VARIETIES

Let us recall the descriptions of Mukai varieties in terms of equations.

3.1. The Grassmannian $G(2, n)$. Let V be a n -dimensional space and $\{y_{i,j}\}_{1 \leq i < j \leq n}$ be the Plücker coordinates in $\bigwedge^2 V$. The Grassmannian $G(2, V)$ is then defined by the 4×4 Pfaffians of the $n \times n$ symmetric matrix involving coordinates:

$$\begin{pmatrix} 0 & y_{12} & y_{13} & \cdots & y_{1n} \\ -y_{12} & 0 & y_{23} & \cdots & y_{2n} \\ & & \cdot & & \cdot \\ -y_{1n} & -y_{2n} & \cdots & y_{(n-1)n} & 0 \end{pmatrix}$$

The tangent space in the point $(1, 0, \dots, 0)$ is then given by $\{y_{ij} = 0 \text{ for } i \geq 3\}$.

3.2. The Orthogonal Grassmannian $SO(5, 10)$. As it was proved for example in [11] the variety $SO(5, 10)$ is given in \mathbb{P}^{14} with coordinates $x_0, x_{12}, \dots, x_{45}, x_{2345}, \dots, x_{1234}$ by the equations:

$$\begin{aligned} 0 &= x_0 x_{2345} + x_{23} x_{45} - x_{24} x_{35} + x_{34} x_{25} \\ 0 &= x_0 x_{1345} + x_{13} x_{45} - x_{14} x_{35} + x_{34} x_{15} \\ 0 &= x_0 x_{1245} + x_{12} x_{45} - x_{14} x_{25} + x_{24} x_{15} \\ 0 &= x_0 x_{1235} + x_{12} x_{35} - x_{13} x_{25} + x_{23} x_{15} \\ 0 &= x_0 x_{1234} + x_{12} x_{34} - x_{13} x_{24} + x_{23} x_{14} \\ (3.1) \quad 0 &= x_{12} x_{1345} + x_{13} x_{1245} + x_{14} x_{1235} + x_{15} x_{1234} \\ 0 &= -x_{12} x_{2345} + x_{23} x_{1245} + x_{24} x_{1235} + x_{25} x_{1234} \\ 0 &= -x_{13} x_{2345} - x_{23} x_{1345} + x_{34} x_{1235} + x_{35} x_{1234} \\ 0 &= -x_{14} x_{2345} - x_{24} x_{1345} - x_{34} x_{1245} + x_{45} x_{1234} \\ 0 &= -x_{15} x_{2345} - x_{25} x_{1345} - x_{35} x_{1245} - x_{45} x_{1235} \end{aligned}$$

The first five equations can be interpreted as defining x_{ijkl} to be the appropriate Pfaffian of the antisymmetric matrix whose upper right half is defined by $(x_{\alpha\beta})_{\alpha < \beta}$, where x_0 is just the homogenizing term, and the five remaining equations are just the standard syzygies between the Pfaffians.

3.3. The Lagrangian Grassmannian $LG(3, 6)$. The variety $LG(3, 6)$ is a non-proper linear section of $G(3, 6) = G(3, W)$ by a 13 dimensional space. It is described in the following way. Let us choose coordinates of \mathbb{P}^{13} and call them $u, z, x_{i,j}, y_{i,j}$, for all $1 \leq i \leq j \leq 3$. Then, $LG(3, 6)$ is isomorphic to the variety given by the standard determinantal equations relating the symmetric matrix

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{1,2} & x_{2,2} & x_{2,3} \\ x_{1,3} & x_{2,3} & x_{3,3} \end{pmatrix}$$

to its adjoint represented by

$$\begin{pmatrix} y_{1,1} & y_{1,2} & y_{1,3} \\ y_{1,2} & y_{2,2} & y_{2,3} \\ y_{1,3} & y_{2,3} & y_{3,3} \end{pmatrix}$$

and it's determinant z , homogenized by u . In fact we may see this as follows. Let $\{x_{ijk}\}_{1 \leq i < j < k \leq 6}$ be the Plucker coordinates in $\wedge^3 W$. Then we can rearrange the equations of $G(3, W)$ to correspond to: $uY = \wedge^2 X, uzI = XY = YX, zX = \wedge^2 Y$, where: $u = x_{123}$,

$$X = \begin{pmatrix} x_{234} & -x_{134} & x_{124} \\ x_{235} & -x_{135} & x_{125} \\ x_{236} & -x_{136} & x_{126} \end{pmatrix}, Y = \begin{pmatrix} x_{156} & -x_{146} & x_{124} \\ x_{256} & -x_{246} & x_{125} \\ x_{356} & -x_{346} & x_{126} \end{pmatrix},$$

$z = x_{456}$ and I is the unit 3×3 matrix. Now considering the Lagrangian Grassmannian with respect to the 2-form $\omega = x_1 \wedge x_4 + x_2 \wedge x_5 + x_3 \wedge x_6$ correspond to requiring that the matrices X and Y are symmetric (for more details see [9]). We shall however also use descriptions of $LG(3, 6)$ given by other generic 2-forms corresponding to different symmetries of the above data.

3.4. The adjoint \mathbb{G}_2 variety. The variety \mathbb{G}_2 is a non-proper linear section of the Grassmannian $G(2, 7) = G(2, V)$ by a 13 dimensional space. It is described in coordinates a, \dots, n by the 4×4 Pfaffians of the matrix

$$M_{G_2} = \begin{pmatrix} 0 & -f & e & g & h & i & a \\ f & 0 & -d & j & k & l & b \\ -e & d & 0 & m & n & -g-k & c \\ -g & -j & -m & 0 & c & -b & d \\ -h & -k & -n & -c & 0 & a & e \\ -i & -l & g+k & b & -a & 0 & f \\ -a & -b & -c & -d & -e & -f & 0 \end{pmatrix}.$$

This can be seen as follows. The condition of being isotropic with respect to a chosen generic 4 form ω can be translated as the dual condition to considering $\{\alpha \in \bigwedge^2 V^* : \alpha \wedge \omega^* = 0\}$, where ω^* is the dual form to ω and V^* is the dual space. In our case $\omega^* = x_1 \wedge x_2 \wedge x_3 + x_4 \wedge x_5 \wedge x_6 + x_1 \wedge x_4 \wedge x_7 + x_2 \wedge x_6 \wedge x_7 + x_3 \wedge x_6 \wedge x_7$. Comparing the wedge product $\alpha \wedge \omega$ to 0 we get 7 equations for the coefficients of α translating to the above symmetry of the matrix.

4. THE PROOFS

Let us now pass to the proofs.

Proof of theorem 2.1. Let S be the variety in \mathbb{P}^{14} given by the equations 3.1. By homogeneity we may assume that we project a section of S from the point $o = (1, 0, \dots, 0)$. Observe also, that we may assume that our linear section is a general linear section of a nodal hyperplane section of S_{10} . The tangent space to S is given by $T_{S,o} = \{x_{2345} = 0, \dots, x_{1234} = 0\}$. The projection S' of S from o is the syzygy variety given by:

$$\begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ -x_{12} & 0 & x_{23} & x_{24} & x_{25} \\ -x_{13} & -x_{23} & 0 & x_{34} & x_{35} \\ -x_{14} & -x_{24} & -x_{34} & 0 & x_{45} \\ -x_{15} & -x_{25} & -x_{35} & -x_{45} & 0 \end{pmatrix} \begin{pmatrix} x_{2345} \\ x_{1345} \\ x_{1245} \\ x_{1235} \\ x_{1234} \end{pmatrix} = 0$$

Let $H = \{ax_{2345} + bx_{1345} + cx_{1245} + dx_{1235} + ex_{1234}\}$ be a hyperplane containing $T_{S,o}$ and let H' be the projection of H from o . Observe that X' , the closure of the image of $X = H \cap S$ by the projection from o , is contained in $H' \cap S'$. Observe moreover that the tangent cone C of X in o is given in $T_{X,o} = T_{S,o}$ by the quadric $Q_1 = \{a(x_{23}x_{45} - x_{24}x_{35} + x_{34}x_{25}) + b(x_{13}x_{45} - x_{14}x_{35} + x_{34}x_{15}) + c(x_{12}x_{45} - x_{14}x_{25} + x_{24}x_{15}) + d(x_{12}x_{35} - x_{13}x_{25} + x_{23}x_{15}) + e(x_{12}x_{34} - x_{13}x_{24} + x_{23}x_{14})\}$ which we assume to be nonsingular as we study only nodal hyperplane sections. Then X' is contained and hence is an irreducible component of $X'' = H' \cap S' \cap \{Q_1 = 0\}$. We shall prove that a general 5-dimensional linear section of X'' is isomorphic to the Mukai variety $G(2, 5) \cap Q$.

Indeed, in a suitable coordinate chart (such that $H = \{x_{2345} = 0\}$) the variety X'' is the intersection of the variety defined by the equations:

$$\begin{pmatrix} x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & x_{23} & x_{24} & x_{25} \\ -x_{23} & 0 & x_{34} & x_{35} \\ -x_{24} & -x_{34} & 0 & x_{45} \\ -x_{25} & -x_{35} & -x_{45} & 0 \end{pmatrix} \begin{pmatrix} x_{1345} \\ x_{1245} \\ x_{1235} \\ x_{1234} \end{pmatrix} = 0$$

with the quadric:

$$x_{23}x_{45} - x_{24}x_{35} + x_{34}x_{25} = 0.$$

We rearrange these equations to get:

$$x_{12}x_{1345} + x_{13}x_{1245} + x_{14}x_{1235} + x_{15}x_{1234} = 0$$

and Pfaffians of the matrix

$$\begin{pmatrix} 0 & x_{1234} & -x_{1235} & x_{1245} & -x_{1345} \\ -x_{1234} & 0 & x_{23} & x_{24} & x_{34} \\ x_{1235} & -x_{23} & 0 & x_{25} & x_{35} \\ -x_{1245} & -x_{24} & -x_{34} & 0 & x_{45} \\ x_{1345} & -x_{25} & -x_{35} & -x_{45} & 0 \end{pmatrix}$$

The first equation defines a cone over a 7 dimensional quadric centered in the space generated by coordinates $x_{23}, x_{24}, x_{25}, x_{34}, x_{35}, x_{45}$ whereas the latter defines a cone over $G(2,5)$ centered in the space generated by coordinates $x_{12}, x_{13}, x_{14}, x_{15}$. A general linear section of dimension 5 of this variety is isomorphic to the Mukai variety M_6^5 representing a generic section of $G(2,5)$ with a quadric. \square

We shall proceed similarly with Theorem 2.2.

Proof. By homogeneity we can assume that we project from the point $o = (1, 0, \dots, 0)$ (where $y_{12} = 1$) lying on the variety G given by Pfaffians of the following matrix:

$$\begin{pmatrix} 0 & y_{12} & y_{13} & y_{14} & y_{15} & y_{16} \\ -y_{12} & 0 & y_{23} & y_{24} & y_{25} & y_{26} \\ -y_{13} & -y_{23} & 0 & y_{34} & y_{35} & y_{36} \\ -y_{14} & -y_{24} & -y_{34} & 0 & y_{45} & y_{46} \\ -y_{15} & -y_{25} & -y_{35} & -y_{45} & 0 & y_{56} \\ -y_{16} & -y_{26} & -y_{36} & -y_{46} & -y_{56} & 0 \end{pmatrix}$$

Observe that the image G' of the projection of G from o is defined by the 9 Pfaffians not involving y_{12} . Recall that the tangent space $T_{G,o} = \{y_i j = 0 \text{ for } i \geq 3\}$. Take a hyperplane $H = \{ay_{34} + by_{35} + cy_{36} + dy_{45} + ey_{46} + fy_{56} = 0\}$ containing $T_{G,o}$. Assuming that $H \cap G$ has a node at o we obtain that the tangent cone at o in $T_{G,o}$ is the quadric defined by $Q_1 = a(y_{13}y_{24} - y_{23}y_{14}) + b(y_{13}y_{25} - y_{23}y_{15}) + c(y_{13}y_{26} - y_{23}y_{16}) + d(y_{14}y_{25} - y_{24}y_{15}) + e(y_{14}y_{26} - y_{24}y_{16}) + f(y_{15}y_{26} - y_{25}y_{16})$, which needs to be smooth. Now, by antisymmetric row and column operations on the matrix and after a suitable change of coordinates we can assume that $H = \{y_{34} + y_{56} = 0\}$ and $Q_1 = y_{13}y_{24} - y_{23}y_{14} - y_{15}y_{26} + y_{25}y_{16}$. The image X' of the projection of $X = G \cap H$ from o is contained in $X'' = G' \cap \{y_{34} - y_{56} = 0\} \cap \{Q_1 = 0\}$. Since X' and X'' have the same

dimension X' is a component of X'' . Now, it remains to prove that X'' is isomorphic to an irreducible linear section of S_{10} .

Writing down the equations of X'' we get:

$$\begin{aligned}
y_{34}^2 - y_{35}y_{46} + y_{36}y_{45} &= 0 \\
y_{13}y_{45} - y_{14}y_{35} + y_{15}y_{34} &= 0 \\
y_{13}y_{46} - y_{14}y_{36} + y_{16}y_{34} &= 0 \\
y_{13}y_{34} - y_{15}y_{36} + y_{16}y_{35} &= 0 \\
y_{14}y_{34} - y_{15}y_{46} + y_{16}y_{45} &= 0 \\
y_{23}y_{45} - y_{24}y_{35} + y_{25}y_{34} &= 0 \\
y_{23}y_{46} - y_{24}y_{36} + y_{26}y_{34} &= 0 \\
y_{23}y_{34} - y_{25}y_{36} + y_{26}y_{35} &= 0 \\
y_{24}y_{34} - y_{25}y_{46} + y_{26}y_{45} &= 0 \\
y_{13}y_{24} - y_{14}y_{23} + y_{15}y_{26} - y_{16}y_{25} &= 0
\end{aligned}$$

We finish the proof by observing that these are exactly the equations 3.1 of S_{10} intersected with the linear space $\{x_0 - x_{2345} = x_{23} = x_{45} = 0\}$ under the identification:

$$\begin{aligned}
&(y_{13}, y_{14}, y_{15}, y_{16}, y_{23}, y_{24}, y_{25}, y_{26}, y_{34}, y_{35}, y_{36}, y_{45}, y_{46}) = \\
&(-x_{12}, x_{13}, x_{1235}, -x_{1234}, x_{1245}, x_{1345}, x_{15}, x_{14}, x_0, -x_{25}, -x_{24}, x_{35}, x_{34}).
\end{aligned}$$

In other terms the equations 3.1 of $OG(5, 10)$ restricted to the space $\{x_0 - x_{2345} = x_{23} = x_{45} = 0\}$ correspond to Pfaffians of the matrix

$$\begin{pmatrix}
0 & \star & -x_{12} & x_{13} & x_{1235} & -x_{1234} \\
& 0 & x_{1245} & x_{1345} & x_{15} & x_{14} \\
& & 0 & x_0 & -x_{25} & -x_{24} \\
& & & 0 & x_{35} & x_{34} \\
& & & & 0 & x_0 \\
& & & & & 0
\end{pmatrix}$$

not involving \star

□

Our proof of theorem 1.1 is the following

Proof. By homogeneity we can assume that we project from the point p with $u = 1$ and with the remaining coordinates 0. Then the tangent space at this point is spanned by all $x_{i,j}$. There are only 4 orbits of the considered action of the symplectic group on \mathbb{P}^13 . The singularities of the hyperplane sections of $LG(3, 6)$ corresponding to points on these orbits are described in [9]. Only points on the open orbit of the dual variety define nodal hyperplane sections of $LG(3, 6)$. We may hence

assume that the nodal section is given by $H = \{y_{2,2} = y_{1,3}\}$. Then, the projection from p of $H \cap LG(3,6)$ is contained in the intersection of the variety defined in H by the equations 3.3 not involving u with the quadric defined by $x_{1,2}x_{2,3} - x_{1,3}x_{2,2} = x_{1,1}x_{3,3} - x_{1,3}^2$. These equations may be rearranged to be the minors of the matrix

$$\begin{pmatrix} 0 & z & y_{1,2} & y_{2,3} & y_{2,2} & y_{1,1} \\ & 0 & y_{2,2} & y_{3,3} & y_{2,3} & y_{1,2} \\ & & 0 & x_{1,2} & x_{1,3} & x_{3,3} \\ & & & 0 & x_{1,1} & x_{2,2} - x_{1,3} \\ & & & & 0 & x_{2,3} \\ & & & & & 0 \end{pmatrix}$$

The minors involving $x_{2,2} - x_{1,3}$ represent then sums of pairs of equations from 3.3. \square

The theorem 2.3 is studied as follows.

Proof. Here we have at least two orbits of nodal sections. The section given by $j = c + f$ is a general one singular at $h = 1$. Indeed we can use the explicit description of the Lie algebra of \mathbb{G}_2 (see [13]) and compute directly the dimension of the stabilizer of the point corresponding via the Killing form to this section and obtain that its projective orbit is of codimension 1. The section given by $e = j$ is also nodal (at $i=1$) but lies in the intersection of the dual variety with the quadric defined by the Killing form. Consider first this special case. Then the image of the projection is given by all Pfaffians of the matrix M_{G_2} not involving i and the quadric $a^2 - gl + f(h + b)$. We check directly that these equations define the same ideal as a codimension 2 linear section of $LG(3,6)$ given by

$$(u, x_{1,1}, x_{1,2}, x_{1,3}, x_{2,2}, x_{2,3}, x_{3,3}, y_{1,1}, y_{1,2}, y_{1,3}, y_{2,2}, y_{2,3}, y_{3,3}, z) = (f, -d, e, b, g, a, l, h + b, -k, d, -e, -c, m, n)$$

In other terms the equations of the projection correspond to the data

$$f, \begin{pmatrix} -d & e & b \\ e & g & a \\ b & a & l \end{pmatrix}, \begin{pmatrix} h + b & -k & d \\ -k & -e & -c \\ d & -c & m \end{pmatrix}, n.$$

In the example corresponding to the general case we project from $h = 1$ and the projection is given by the Pfaffians of M_{G_2} not involving h and the quadric $a(a - n - d) + gl - f(i + e + b) = 0$. We can rearrange the

equations to be defined by the following data

$$a, \begin{pmatrix} -d & b & c \\ g & f & e \\ i+e+b & l & k \end{pmatrix}, \begin{pmatrix} -b & a-n-d & -g-k \\ -c-f & g+k & d-l \\ d & m & c+f \end{pmatrix}, -m-b.$$

The latter is a proper codimension 2 linear section of the Lagrangian Grassmannian corresponding to the form

$$\omega = x_1 \wedge x_6 - x_3 \wedge x_5 + x_2 \wedge x_4 - x_1 \wedge x_2.$$

□

Problem 4.1. It is an interesting question to find a direct, intrinsic way to describe above constructions. L. Manivel private communication showed us such a way to prove Theorem 2.2.

5. APPLICATIONS

In this section we explain how the results of the paper may be used to describe some moduli spaces of bundles on curves, K3 surfaces or Fano 3-folds. We follow the idea presented in [9]. More precisely, let X be a generic curve of genus $g \leq 9$ (or K3 surface, Fano 3 fold of genus $g \leq 10$). Then X is a linear section of the variety M_g and each nodal linear space H containing X with node outside X induces an embedding $\pi_H : X \rightarrow M_{g-1}$.

Proposition 5.1. *Two different linear spaces H of maximal dimension define different embeddings.*

Proof. The proof contained in [9] is valid in all cases. Indeed, assume we have H_1 and H_2 inducing isomorphic embeddings. Then the intersections of the linear spans of their images with M_{g-1} are projectively isomorphic. Hence their preimages by the projections are also projectively isomorphic, but these are just the sections of M_g by linear spans of X and the projecting nodes. By Mukai's result (see [6]) these are isomorphic only if they lie in the same orbit of the appropriate group action. This would induce an automorphism of X which by genericity would have to be trivial and in consequence the automorphism of M_g inducing it would also need to be trivial. □

Remark 5.2. The results of this paper may be used to describe the space of polarized K3 surfaces of genus $g-1$ containing a chosen general curve of genus $g \leq 9$ in its canonical embedding. One might observe that such a general K3 surface S has Picard number 2 generated by the hyperplane H and a conic Q . The curve of genus g lies then in the system $|H + Q|$, which induces a morphism from S to M_g contracting

Q to a node. Its image is a nodal linear section of M_g hence is a linear section of a nodal hyperplane section. It follows that it appears in our construction. Concluding the space of K3 surfaces genus $g - 1$ containing a chosen general curve of genus $g \leq 9$ in its canonical embedding is isomorphic to a linear section of the dual variety to M_g by the dual linear space to the span of the chosen curve. The same is valid for Fano threefolds containing a K3 surface of genus smaller by one.

Let us now consider the situation for $g = 7$.

Lemma 5.3. *Let X be a general curve of genus $g = 7$ embedded as a linear section of $SO(5, 10)$. Consider a nodal linear space H of dimension 7 containing X and denote by p its node. Let $\pi : X \rightarrow G(2, 5) \cap Q$ be the projection from p and let E be the pullback of the universal quotient bundle on $G(2, 5)$. Then E is stable on X .*

Proof. By assumption X has no g_4^1 . Observe first that by construction E is a rank 2 bundle on X such that $c_1(E) = K_X$, $h^0(E) = 5$. By contradiction assume that there is a line bundle L on X of degree $d \geq 4$ which is a subbundle of E . We get the exact sequence:

$$0 \longrightarrow L \longrightarrow E \longrightarrow M \longrightarrow 0,$$

where $M = K_X \otimes L^*$. It follows that $h^0(L) + h^0(M) \geq h^0(E) = 5$. On the other hand by the Riemann-Roch formula we have $h^0(L) - h^0(M) = d - 6 \geq -2$. Hence $h^0(L) \geq 2$, which contradicts the assumption on C . \square

The following well known corollaries (see [7],[8]) follow.

Corollary 5.4. Let C be a general curve of genus 7. Then, the Brill Noether locus $M_C(2, K_X, 5)$ parameterizing rank 2 vector bundles with canonical determinant and with 5 global sections is a Fano variety of genus 7 obtained as the dual linear section of $SO(5, 10)$.

Proof. By dimension count it is a direct consequence of Proposition 5.1, Lemma 5.3 and the fact that the orthogonal Grassmannian is self-dual. \square

We also recover in the same way the result concerning K3 surfaces.

Corollary 5.5. Let (S, L) be a BN-general polarized K3 surface of genus 7. Then, the K3 surface $M = M_C(2, K_X, 3)$ parameterizing rank 2 vector bundles with canonical determinant and second Chern class of degree 3 is isomorphic to the dual linear section of $SO(5, 10)$.

Let us pass to the case $g = 10$

Lemma 5.6. *Let X be a curve of genus $g = 10$ embedded as a general linear section of \mathbb{G}_2 . Consider a nodal linear space H of dimension 10 containing X and denote by p its node. Let $\pi : X \rightarrow LG(3, V)$, where V is a vector space of dimension 6, be the projection from p and let E be the pullback of the universal quotient bundle on $G(3, V)$. Then E is stable on X .*

Proof. We follow the idea of [9]. By assumption X has no g_6^1 . Observe first that by construction E is a rank 3 bundle on X such that $c_1(E) = K_X$, $h^0(E) = 6$. We have two possibilities to destabilize E . Either there is a subbundle of E of rank 1 and degree ≥ 6 or a rank 2 subbundle of degree ≥ 12 . We consider the two cases separately.

Assume first that there is a line bundle L on X of degree $d \geq 6$ which is a subbundle of E . We get the exact sequence:

$$0 \longrightarrow L \longrightarrow E \longrightarrow F \longrightarrow 0,$$

with $c_1(F) = K_X \otimes L^*$. It follows that $h^0(L) + h^0(F) \geq h^0(E) = 6$. On the other hand by the Riemann-Roch formula we have $h^0(L) - h^0(K_X \otimes L^*) = d - 9 \geq -3$. We claim that $h^0(L) \geq 1$. Indeed if L had no sections then the map $H^0(E) \rightarrow H^0(F)$ would be an embedding. Hence the bundle F would be globally generated by 6 sections. We could then consider a nonvanishing section of F . Its wedge products with the remaining generators would give 5 linearly independent sections of $c_1(F)$. The latter contradicts the Riemann-Roch formula. Hence $h^0(L) = 1$ (by assumption $h^0(L) \leq 1$) and its section defines a hyperplane in $\mathbb{P}(H^0(E))$ which contains d fibers of the natural image of $\mathbb{P}(E)$ in $\mathbb{P}(H^0(E))$. It follows that X intersects a $G(3, W)$ (for some hyperplane $W \subset V$) in $d \geq 6$ points. We observe now that the linear span of X intersects $G(3, V)$ in a K3 surface S . The surface S being irreducible meets $G(3, W)$ in at most a curve, hence the d points of $X \cap G(3, W)$ span at most a \mathbb{P}^4 . But X admits no g_6^1 , hence the span of these points is a \mathbb{P}^5 leading to a contradiction.

Assume now that there is a rank 2 bundle F on X of degree $d \geq 12$ which is a subbundle of E . We get the exact sequence:

$$0 \longrightarrow F \longrightarrow E \longrightarrow L \longrightarrow 0.$$

Then $L = K_X \otimes c_1(F)^*$ has at most one section by assumption on X . Hence, $h^0(F) \geq 5$. But we have proved above that E does not admit any line subbundle with 2 sections (as by assumption on X it would have degree > 6), hence neither does F . It follows that the projectivisation of the kernel of the map $\bigwedge^2 H^0(F) \rightarrow H^0(\bigwedge^2 F) = H^0(c_1(F))$ does not meet the Grassmannian $G(2, H^0(F))$. Finally $h^0(c_1(F)) \geq 7$ which contradicts the Riemann-Roch for L . \square

Remark 5.7. The bundles considered in Lemma 5.6 were already constructed in a different context by Kuznetsov in [4].

Corollary 5.8. Let C be a general curve of genus 10 contained in a K3 surface. Then, the non-abelian Brill-Noether locus $M(3, K_C, 6)$ contains a surface M_L of degree 6 which is the linear section of the dual variety of \mathbb{G}_2 by the dual space to the span of C . The bundles corresponding to points of this surface correspond to embeddings of C into $LG(3, 6)$.

Proof. It follows from the fact that the variety dual to G_2 is a hypersurface of degree 6 (see [1]) and the fact that our G_2 is the homogeneous space of the Adjoint representation of the simple Lie Group G_2). Repeating the reasoning contained in Remark 5.2 we prove that in this construction we obtain all embeddings of C into $LG(3, 6)$. \square

We can view the surface as a non-abelian Brill-Noether locus of all quotient Lagrangian bundles, but first we need to prove that such moduli space indeed exists. Let us make a general argument. Let C be a curve of genus g . Let $E = \mathcal{O}_C^{2n}$, we say that a quotient bundle L of E of rank n on C with $c_1(L) = K_C$ is Lagrangian if the image of the map associated to this bundle to the Grassmannian $G(n, 2n)$ is contained in some $LG(n, 2n)$. This is equivalent to saying that there is an exact sequence:

$$0 \longrightarrow L^* \longrightarrow E \longrightarrow L \longrightarrow 0$$

and a nondegenerate two form ω on $H^0(E)$ such that all fibers of L^* are isotropic with respect to ω . It follows that for a Lagrangian bundle the kernel of the map $\phi_L: H^0(\bigwedge^2(E)) \longrightarrow H^0(\bigwedge^2(L))$ is non-empty. The set of bundles with this property may be given the structure of a scheme as in [7] because it may be regarded as the degeneracy locus of a map of bundles. This scheme may be saturated with respect to the rank of the map. Now the set of Lagrangian bundles is an open subset of any component of the scheme of bundles for which ϕ has non-empty kernel of chosen dimension. It follows that there is a natural structure of scheme on the set of Lagrangian bundles. If C is a curve of genus 10 and $n = 3$ we have $h^0(C, \bigwedge^2(E)) = h^0(C, \bigwedge^2(L)) = 15$. Hence our moduli is expected to be a hypersurface in the Brill-Noether locus $M(3, K_C, 6)$ and in fact it is such by Corollary 5.8.

Corollary 5.9. Let (X, L) be a polarized K3 surface of genus 10. The space $M_X(3, L, 3)$ contains a smooth plane curve of degree 6 which is the linear section of the dual variety of \mathbb{G}_2 by the dual space to the span of X . The bundles corresponding to points of this curve correspond to embeddings of X into $LG(3, 6)$.

Proof. The only thing that remains to be proved is the smoothness of the sextic which follows from genericity and the fact that the singular set of the variety dual to G_2 is of codimension 2. \square

Corollary 5.10. The moduli space $M_X(3, L, 3)$ is a smooth K3 surface isomorphic to a double cover of \mathbb{P}^2 branched over a sextic.

Proof. By [5] the moduli space $M_X(3, L, 3)$ is a smooth K3 surface. Let C be the plane sextic curve on the K3 surface $M_X(3, L, 3)$. Then C is of genus 10 and admits a g_6^2 . By Green and Lazarsfeld theorem the g_6^2 extends to the K3 surface giving a map to \mathbb{P}^2 . \square

For a more detailed description of the above Moduli space see [3].

Remark 5.11. By considering compositions of the studied projections we obtain different subvarieties of suitable Moduli spaces or Brill-Noether loci.

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